

Integrable hyperbolic equations of sin-Gordon type.

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Abstract

A complete list of nonlinear one-field hyperbolic equations having generalized integrable x- and y-symmetries of the third order is presented. The list includes both sin-Gordon type equations and equations linearizable by differential substitutions.

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1 Introduction

The symmetry approach to classification of integrable PDEs (see surveys [1–3] and references there) is based on the existence of higher infinitesimal symmetries and/or conservation laws for integrable equations. This approach especially efficient for evolution equations with one spatial variable. In particular, all integrable equations of the form

$$u_t = u_3 + F(u_2, u_1, u), \quad u_i = \frac{\partial^i u}{\partial x^i} \quad (1.1)$$

were found in [4, 5]. The following list of integrable equations

List 1:

$$u_t = u_{xxx} + uu_x, \quad (1.2)$$

$$u_t = u_{xxx} + u^2 u_x, \quad (1.3)$$

$$u_t = u_{xxx} + u_x^2, \quad (1.4)$$

$$u_t = u_{xxx} - \frac{1}{2}u_x^3 + (c_1 e^{2u} + c_2 e^{-2u})u_x, \quad (1.5)$$

$$u_t = u_{xxx} - \frac{3u_x u_{xx}^2}{2(u_x^2 + 1)} + c_1(u_x^2 + 1)^{3/2} + c_2 u_x^3, \quad (1.6)$$

$$u_t = u_{xxx} - \frac{3u_x u_{xx}^2}{2(u_x^2 + 1)} - \frac{3}{2}\wp(u)u_x(u_x^2 + 1), \quad (1.7)$$

$$u_t = u_{xxx} - \frac{3u_{xx}^2}{2u_x} + \frac{3}{2u_x} - \frac{3}{2}\wp(u)u_x^3, \quad (1.8)$$

$$u_t = u_{xxx} - \frac{3u_{xx}^2}{2u_x}, \quad (1.9)$$

$$u_t = u_{xxx} - \frac{3u_{xx}^2}{4u_x} + c_1 u_x^{3/2} + c_2 u_x^2, \quad c_1 \neq 0 \text{ or } c_2 \neq 0, \quad (1.10)$$

$$u_t = u_{xxx} - \frac{3u_{xx}^2}{4u_x} + cu, \quad (1.11)$$

$$u_t = u_{xxx} - \frac{3}{4} \frac{u_{xx}^2}{u_x + 1} + 3u_{xx}u^{-1}(\sqrt{u_x + 1} - u_x - 1) - 6u^{-2}u_x(u_x + 1)^{3/2} + 3u^{-2}u_x(u_x + 1)(u_x + 2), \quad (1.12)$$

$$u_t = u_{xxx} - \frac{3}{4} \frac{u_{xx}^2}{u_x + 1} - 3 \frac{u_{xx}(u_x + 1) \cosh u}{\sinh u} + 3 \frac{u_{xx}\sqrt{u_x + 1}}{\sinh u} - 6 \frac{u_x(u_x + 1)^{3/2} \cosh u}{\sinh^2 u} + 3 \frac{u_x(u_x + 1)(u_x + 2)}{\sinh^2 u} + u_x^2(u_x + 3), \quad (1.13)$$

$$u_t = u_{xxx} + 3u^2u_{xx} + 3u^4u_x + 9uu_x^2, \quad (1.14)$$

$$u_t = u_{xxx} + 3uu_{xx} + 3u^2u_x + 3u_x^2, \quad (1.15)$$

$$u_t = u_{xxx}. \quad (1.16)$$

can be derived from [4]. Here $(\wp')^2 = 4\wp^3 - g_2\wp - g_3$, and k, c, c_1, c_2, g_2, g_3 are arbitrary constants. Equations (1.2)–(1.10) are integrable by the inverse scattering method whereas (1.11)–(1.15) are linearizable (S and C -integrable in the terminology by F. Calogero). The above list is complete up to transformations of the form

$$u \rightarrow \phi(u); \quad t \rightarrow t, \quad x \rightarrow x + ct; \quad x \rightarrow \alpha x, \quad t \rightarrow \beta t, \quad u \rightarrow \lambda u; \quad u \rightarrow u + \gamma x + \delta t. \quad (1.17)$$

The latter transformation preserves the form (1.1) only for equations with $\frac{\partial F}{\partial u} = 0$. Moreover the linear equations admit the transformation:

$$u \rightarrow u \exp(\alpha x + \beta t). \quad (1.18)$$

Since the symmetry approach is pure algebraic, the function ϕ and the constants $c, \alpha, \beta, \lambda, \gamma$ and δ supposed to be complex-valued. Thus, we do not distinguish equations $u_t = u_{xxx} - u_x^3$ and $u_t = u_{xxx} + u_x^3$ and so on.

For scalar hyperbolic equations of the form

$$u_{xy} = \Psi(u, u_x, u_y) \quad (1.19)$$

the symmetry approach postulates the existence of both x -symmetries

$$u_t = A(u, u_x, u_{xx}, \dots), \quad (1.20)$$

and y -symmetries

$$u_t = B(u, u_y, u_{yy}, \dots). \quad (1.21)$$

Two equations (1.19) are called *equivalent* if they are related by transformations of the form

$$x \leftrightarrow y; \quad u \rightarrow \phi(u); \quad x \rightarrow \alpha x, \quad y \rightarrow \beta y, \quad u \rightarrow \lambda u; \quad u \rightarrow u + \gamma x + \delta y, \quad (1.22)$$

and for the linear equations by

$$u \rightarrow u \exp(\alpha x + \beta y); \quad u \rightarrow u + c x y. \quad (1.23)$$

Here, in general, the function ϕ and the constants supposed to be complex-valued.

For the famous integrable sin-Gordon¹ equation

$$u_{xy} = c_1 e^u + c_2 e^{-u} \quad (1.24)$$

the simplest x and y -symmetries are given by

$$u_t = u_{xxx} - \frac{1}{2}u_x^3, \quad u_t = u_{yyy} - \frac{1}{2}u_y^3.$$

These evolution equations are integrable themselves (a special case of equation (1.5)).

The general higher symmetry classification for equations (1.19) turns out to be very complicated problem, which is not solved till now. Some important special results have been obtained in [6–8]. In general, all three functions Ψ, A, B should be found from the compatibility conditions for equations (1.19), (1.20) and (1.19), (1.21). However, if the functions A and B are somehow fixed, then it is not difficult to verify whether the corresponding function Ψ exists or not and to find it.

To describe all integrable equations (1.19) of the sin-Gordon type, we assume (see Discussion) that both symmetries (1.20) and (1.21) are *integrable* equations of the form

$$u_t = u_{xxx} + F(u, u_x, u_{xx}), \quad u_t = u_{yyy} + G(u, u_y, u_{yy}). \quad (1.25)$$

It turns out that taken for x -symmetry one of the equations of the List 1, one can easily find the corresponding equations (1.19) having y -symmetries or can prove that such equations do not exist. In the Section 2 we present all integrable hyperbolic equations thus obtained.

The hyperbolic equations can be separated by presence or absence of x and y -integrals (see Discussion). Consider, for instance, the Liouville equation

$$u_{xy} = e^u.$$

¹We do not distinguish sin-Gordon and sinh-Gordon equations

It is easy to verify that the function

$$P = u_{xx} - \frac{1}{2}u_x^2$$

does not depend on y (i.e. is a function depending on x only) for any solution $u(x, y)$ of the Liouville equation. Analogously, the function

$$Q = u_{yy} - \frac{1}{2}u_y^2$$

does not depend on x on the solutions of the Liouville equation.

A function $w(x, y, u, u_y, u_{yy}, \dots)$ that does not depend on x for any solution of (1.19) is called *x-integral*. Similarly the *y-integrals* are defined. An equation of the form (1.19) is called equation of the Liouville type (or Darboux integrable equation), if the equation possesses both nontrivial *x*- and *y*-integrals. Some of integrable hyperbolic equations found in Section 2 are equations of the Liouville type.

In contrast, the sin-Gordon equation (1.24) has no *x*- and *y*-integrals for generic values of the constants c_i . There are two types of such equations. Equations of the first type can be reduced to the linear Klein-Gordon equation $u_{xy} = cu$ by differential substitutions. If an equation with the third order symmetries has no integrals and linearizing substitutions, we call it *equation of sin-Gordon type*. The following equations from the lists of Section 2 are equations of this kind:

$$u_{xy} = c_1 e^u + c_2 e^{-u}; \quad (1.26)$$

$$u_{xy} = f(u) \sqrt{u_x^2 + 1}, \quad f'' = cf; \quad (1.27)$$

$$u_{xy} = \sqrt{u_x} \sqrt{u_y^2 + 1}; \quad (1.28)$$

$$u_{xy} = \sqrt{\wp(u) - \mu} \sqrt{u_x^2 + 1} \sqrt{u_y^2 + 1}. \quad (1.29)$$

Here $(\wp')^2 = 4\wp^3 - g_2\wp - g_3$, $4\mu^3 - g_2\mu - g_3 = 0$ and $c, c_1, c_2, a, \mu, g_2, g_3$ are constants. Equations (1.27), (1.29) are known. These equations are related to equation (1.26) via differential substitutions [8, 10]. Equation (1.28) is probably new. The corresponding differential substitution is given by

$$v_{xy} = \frac{1}{2} \cosh v, \quad u_{xy} = \sqrt{u_x} \sqrt{u_y^2 + 1}, \quad v = \ln(u_y + \sqrt{u_y^2 + 1}).$$

2 Hyperbolic equations with third order symmetries

Theorem 1. *Suppose both *x*- and *y*-symmetry of a hyperbolic equation of the form (1.19) belong to the list (1.2)–(1.16) up to transformations (1.17), (1.18). Then this equation belongs*

to the following list:

$$u_{xy} = f(u)\sqrt{u_x^2 + 1}, \quad f'' = cf, \quad (2.1)$$

$$u_{xy} = ae^u + be^{-u}, \quad (2.2)$$

$$u_{xy} = \sqrt{u_x}\sqrt{u_y^2 + 1}, \quad (2.3)$$

$$u_{xy} = \sqrt{u_x^2 + 1}\sqrt{u_y^2 + 1}, \quad (2.4)$$

$$u_{xy} = \sqrt{\wp(u) - \mu}\sqrt{u_x^2 + 1}\sqrt{u_y^2 + a}, \quad (2.5)$$

$$u_{xy} = 2uu_x, \quad (2.6)$$

$$u_{xy} = 2u_x\sqrt{u_y}, \quad (2.7)$$

$$u_{xy} = u_x\sqrt{u_y^2 + 1}. \quad (2.8)$$

$$u_{xy} = \sqrt{u_x u_y}, \quad (2.9)$$

$$u_{xy} = \frac{u_x(u_y + a)}{u}, \quad a \neq 0, \quad (2.10)$$

$$u_{xy} = (ae^u + be^{-u})u_x, \quad (2.11)$$

$$u_{xy} = u_y \eta \sinh^{-1} u (\eta e^u - 1), \quad (2.12)$$

$$u_{xy} = \frac{2u_y \eta}{\sinh u} (\eta \cosh u - 1), \quad (2.13)$$

$$u_{xy} = \frac{2\xi \eta}{\sinh u} ((\xi \eta + 1) \cosh u - \xi - \eta), \quad (2.14)$$

$$u_{xy} = u^{-1} u_y \eta (\eta - 1) + c u \eta (\eta + 1), \quad (2.15)$$

$$u_{xy} = 2u^{-1} u_y \eta (\eta - 1), \quad (2.16)$$

$$u_{xy} = 2u^{-1} \xi \eta (\xi - 1)(\eta - 1), \quad (2.17)$$

$$u_{xy} = u^{-1} u_x u_y - 2u^2 u_y, \quad (2.18)$$

$$u_{xy} = u^{-1} u_x (u_y + a) - u u_y \quad (2.19)$$

$$u_{xy} = \sqrt{u_y} + a u_y, \quad (2.20)$$

$$u_{xy} = cu, \quad (2.21)$$

up to transformations (1.22), (1.23). Here \wp is the Weierstrass function: $(\wp')^2 = 4\wp^3 - g_2\wp - g_3$, $4\mu^3 - g_2\mu - g_3 = 0$, $\xi = \sqrt{u_y + 1}$, $\eta = \sqrt{u_x + 1}$; a, b, c, μ, g_2, g_3 are constants.

Proof. If (1.25) is an x -symmetry for (1.19), then

$$\frac{d^2}{dx dy} (u_{xxx} + F) = \frac{\partial \Psi}{\partial u_x} \frac{d}{dx} (u_{xxx} + F) + \frac{\partial \Psi}{\partial u_y} \frac{d}{dy} (u_{xxx} + F) + \frac{\partial \Psi}{\partial u} (u_{xxx} + F). \quad (2.22)$$

Eliminating all mixed derivatives in virtue of (1.19), we arrive at a defining relation, which has to be fulfilled identically with respect to the variables $u, u_y, u_x, u_{xx}, u_{xxx}$. Comparing the coefficients at u_{xxx} in this relation, we get

$$\frac{d}{dy} \frac{\partial F}{\partial u_{xx}} + 3 \frac{d}{dx} \frac{\partial \Psi}{\partial u_x} = 0. \quad (2.23)$$

If some equation from the list (1.2)–(1.16) is taken for the x -symmetry then the function F is known and the defining relation can also be splitted with respect to u_{xx} .

For example, let equation (1.7) be an x -symmetry for (1.19). Then the u_{xx} -splitting of (2.23) gives rise to:

$$\begin{aligned} (u_x^2 + 1)^2 \frac{\partial^2 \Psi}{\partial u_x^2} - u_x(u_x^2 + 1) \frac{\partial \Psi}{\partial u_x} + (u_x^2 - 1)\Psi &= 0, \\ (u_x^2 + 1) \left(\Psi \frac{\partial^2 \Psi}{\partial u_x \partial u_y} + u_x \frac{\partial^2 \Psi}{\partial u \partial u_x} \right) - u_x^2 \frac{\partial \Psi}{\partial u} - u_x \Psi \frac{\partial \Psi}{\partial u_y} &= 0. \end{aligned}$$

The general solution of this system is given by

$$\Psi = \sqrt{u_x^2 + 1} \left(g(u, u_y) + C \ln(u_x + \sqrt{u_x^2 + 1}) \right).$$

Substituting this expression into (2.22) and finding the coefficient at u_{xx}^3 , we obtain $C = 0$ and therefore

$$\Psi = g(u, u_y) \sqrt{u_x^2 + 1}. \quad (2.24)$$

Splitting (2.22) with respect to u_{xx} and u_x , we obtain that (2.22) is equivalent to a system consisting of (2.24) and equations

$$\begin{aligned} g \frac{\partial^2 g}{\partial u \partial u_y} - \frac{\partial g}{\partial u} \frac{\partial g}{\partial u_y} &= 0, \quad \wp'(u) u_y = 2 \frac{\partial g}{\partial u} \frac{\partial g}{\partial u_y}, \\ g^2 \frac{\partial^2 g}{\partial u_y^2} + g \left(\frac{\partial g}{\partial u_y} \right)^2 - 3g\wp + \frac{\partial^2 g}{\partial u^2} &= 0, \end{aligned} \quad (2.25)$$

where $(\wp')^2 = 4\wp^3 - g_2\wp - g_3$. Since $\wp' \neq 0$ we have $g_u \neq 0$ and $g_{u_y} \neq 0$. It follows from the first two equations (2.25) that $g = \sqrt{\wp(u) - \mu} \sqrt{u_y^2 + a}$, where μ and a are constants of integration. The third equation is equivalent to the algebraic equation $4\mu^3 - g_2\mu - g_3 = 0$ for μ . Thus, we get equation (2.5).

To prove the theorem we perform similar computations for each equation from the list (1.2)–(1.15) taken for x -symmetry. For equations (1.2), (1.4) and (1.8) the corresponding hyperbolic equation does not exist. In contrast, equation (1.12) is an x -symmetry for several different hyperbolic equations. Indeed, in this case calculating the coefficient at u_{xx} in (2.23), we get

$$2(u_x + 1)^2 \frac{\partial^2 \Psi}{\partial u_x^2} - (u_x + 1) \frac{\partial \Psi}{\partial u_x} + \Psi = 0,$$

which implies $\Psi = f_1(u, u_y)(u_x + 1) + f_2(u, u_y)\sqrt{u_x + 1}$. Substituting this into (2.23), we obtain

$$\begin{aligned} \left(u \frac{\partial f_1}{\partial u_y} - 1 \right) \left(u^2 f_1 \frac{\partial f_1}{\partial u_y} - 3u f_1 + 2u_y \right) &= 0, \\ u^2 \left(f_1 \frac{\partial f_1}{\partial u_y} + \frac{\partial f_1}{\partial u} \right) - 2u f_1 + 2u_y &= 0, \\ f_2 &= 2f_1 - \frac{2}{u} u_y - u f_1 \frac{\partial f_1}{\partial u_y}. \end{aligned}$$

If the first factor in the first equation is equal to zero, we arrive at (2.15). If the second factor equals zero, then we get

$$f_1 = \frac{2u_y \sqrt{au_y + 1}}{u(1 + \sqrt{au_y + 1})},$$

where a is a constant. The case $a \neq 0$ corresponds to (2.17), while $a = 0$ leads to equation (2.15) with $c = 0$. The limit $a \rightarrow \infty$ gives us equation (2.16).

The computations for remaining x -symmetries from the list except for the Swartz-KdV equation (1.9) are very similar and we do not demonstrate them here.

Consider the Swartz-KdV equation (1.9). Equation (1.9) is exceptional because there is a wide class of hyperbolic equations with this x -symmetry. Not all equations from this class are integrable and we derive those of them that have y -symmetries.

It is easy to verify that equation

$$u_{xy} = f(u, u_y)u_x \quad (2.26)$$

has the following symmetry

$$u_t = u_{xxx} - \frac{3u_{xx}^2}{2u_x} + q(u)u_x^3, \quad (2.27)$$

where

$$\left(\frac{\partial}{\partial u} + f \frac{\partial}{\partial u_y} \right)^2 f + 2qf + q'u_y = 0. \quad (2.28)$$

The function $q(u)$ can be eliminated by an appropriate transformation $u \rightarrow \varphi(u)$, but we prefer to use transformations of this type for bringing the y -symmetry to one of equations (1.2)–(1.15). Here and in the sequel we make the transformation $x \leftrightarrow y$ in formulas (1.2)–(1.15) as well as in other formulas we need.

Any of y -symmetries has the form

$$u_t = u_3 + A_2(u, u_1)u_2^2 + A_1(u, u_1)u_2 + A_0(u, u_1), \quad u_n = \frac{\partial^n u}{\partial u_y^n}.$$

Equation (2.23) with $x \leftrightarrow y$ is equivalent to

$$\begin{aligned} 3 \frac{\partial^2 f}{\partial u_y^2} + 2 \frac{\partial(fA_2)}{\partial u_y} + 2 \frac{\partial A_2}{\partial u} &= 0, \\ 3u_y \frac{\partial^2 f}{\partial u \partial u_y} + 3f \frac{\partial f}{\partial u_y} + 2A_2u_y \frac{\partial f}{\partial u} + f \frac{\partial A_1}{\partial u_y} + 2A_2f^2 + \frac{\partial A_1}{\partial u} &= 0. \end{aligned} \quad (2.29)$$

Equations (2.28) and (2.22) give rise to additional restrictions for the functions f and q .

For symmetries (1.2)–(1.5) we have $A_1 = A_2 = 0$ and equations (2.29) imply $f = u_y g(u) + h(u)$, $gh = 0$, $g' + g^2 = 0$. In the case $g \neq 0$ we get (2.10) with $a = 0$. For $g = 0$ it

follows from (2.28) that $q' = 0$ and $f'' + 2qf = 0$. If $q \neq 0$, then without loss of generality we take $q = -\frac{1}{2}$ and arrive at equation (2.11). In the case $g = 0, q = 0$ we get equation (2.6).

For symmetries (1.6) and (1.7) $A_1 = 0, A_2 = -3/2 u_y(u_y^2 + 1)^{-1}$. It follows from (2.28) and (2.29) that $f = h(u)u_x\sqrt{u_y^2 + 1}, h'' = 2h(h^2 + c_0), q = c_0 - 3/2 h^2$. If $h' = 0$, then we put $h = 1$ and obtain equation (2.8). In the case $h' \neq 0$ we get $h = \sqrt{\wp - \mu}, q = -3/2\wp$. The hyperbolic equation is given by (2.5) with $x \leftrightarrow y$ and $a = 0$.

For symmetries (1.8), (1.9) $A_2 = -\frac{3}{2}u_y^{-1}, A_1 = 0$. It follows from (2.29) that $f = g(u)u_y$. So, we obtain the equation $u_{xy} = g(u)u_xu_y$. Both x - and y -symmetries of the equation take the form (2.27), where

$$q = C \exp\left(-2 \int g(u) du\right) - g' - \frac{1}{2}g^2.$$

The equation is equivalent to the D' Alembert equation $u_{xy} = 0$ under the following transformation:

$$\bar{u} = \int du \exp\left(-\int g(u) du\right).$$

For symmetries (1.10) and (1.11) $A_2 = -\frac{3}{4}u_y^{-1}, A_1 = 0$. It follows from (2.29), (2.28) that $f = g(u)u_y + C\sqrt{u_y}, gC = 0, qC = 0, g' + g^2 = 0, q' + 2qg = 0$. If $C \neq 0$, then $q = g = 0$. Taking $C = 2$, we get (2.7). If $C = 0$, then $g = u^{-1}, q = c_0u^{-2}$, and we obtain (2.10) with $a = 0$.

If the y -symmetry has the form (1.12), then it follows from (2.28), (2.29) that $f = ku^{-1}(u_y + 1 - \sqrt{u_y + 1}), (k-1)(k-2) = 0, q = 3(2-k)/(8u^2)$. If $k = 1$ we get (2.15) with $x \leftrightarrow y$. The case $k = 2$ leads to (2.16).

In the case of y -symmetry (1.13) the system of equations (2.28), (2.29) has two solutions corresponding to equations (2.12), (2.13) with $x \leftrightarrow y$.

Symmetry (1.14) gives rise to equation (2.18) with $x \leftrightarrow y$.

Symmetry (1.15) corresponds to the following equation

$$u_{xy} = \frac{u_x u_y}{u + a} - (u + a)u_x.$$

The shift $u \rightarrow u - a$ brings it to a special case of equation (2.19).

Considering the linear x -symmetry (1.16), we obtain equation (2.10) with arbitrary parameter a , equation (2.21), and

$$u_{xy} = a u_x + f(u_y - a u), \quad (2.30)$$

where f satisfies some nonlinear third order ODE. The requirement of the existence of a y -symmetry leads to (2.20).

More detailed information of each equation of the list (2.1)–(2.21) can be found in Appendix 1.

Discussion.

The hyperbolic equations of the form (1.19) having both x and y -integrals were described in [8]. In particular, it was shown that any such equation possesses both x and y higher symmetries depending on arbitrary functions. Although not all of these symmetries are integrable, usually some integrable symmetries exist for such equations.

There are integrable equations having only y -integrals (or only x -integrals). An example of such equation is given by (2.5), where $a = 0$. Namely, the equation

$$u_{xy} = \xi'(u)u_y\sqrt{u_x^2 + 1}, \quad (2.31)$$

where $\xi'(u) = \sqrt{\wp - \mu}$, has the following first order y -integral

$$I = (u_x + \sqrt{u_x^2 + 1}) e^{-\xi}$$

and has no x -integrals for the generic Weierstrass function \wp . Notice that the same formula gives an y -integral for (2.31) with arbitrary function ξ .

In some sense equations (1.19) having integrals can be reduced to ODEs. If we are looking for equations (1.19) integrable by the inverse scattering transform method, we should concentrate on integrable equations (1.19) without integrals. There are two classes of such equations. The first one consists of the Klein-Gordon equation $u_{xy} = cu$, $c \neq 0$ and equations related through differential substitutions to the Klein-Gordon equation. The symmetries for such equations are C -integrable (in terminology by F. Calogero). The second class of hyperbolic integrable equations having no integrals contains equations that can not be reduced to a linear form by differential substitutions. This the most interesting class consists of hyperbolic equations admitting only S -integrable higher symmetries. Such equations can be regarded as *S-integrable* hyperbolic equations.

For the first glance the anzats (1.25) seems to be very restrictive if we want to describe all S -integrable equations (1.19). The first question is: why only third order equations are taken for symmetries? We can justify it in the following way. All known S -integrable hierarchies of evolution equations (1.20) contain either a third order or a fifth order equation. For polynomial equations this is not an observation but a rigorous statement [9]. That is why it is enough to consider hyperbolic equations with symmetries of third order (sin-Gordon type equations) and hyperbolic equations with fifth order symmetries (Tzitzeica type equations).

The following Tzitzeica type S -integrable equations are known [8, 11]:

$$u_{xy} = c_1 e^u + c_2 e^{-2u}, \quad (2.32)$$

$$u_{xy} = S(u) f(u_x) g(u_y), \quad (2.33)$$

$$u_{xy} = \frac{\omega' + 3c}{\omega(u)} f(u_x) g(u_y), \quad (2.34)$$

$$u_{xy} = h(u) g(u_y), \quad h'' = 0, \quad (2.35)$$

where

$$\begin{aligned} (f + 2u_x)^2 (u_x - f) &= 1, \quad (g + 2u_y)^2 (u_y - g) = 1, \\ (S' - 2S^2)^2 (S' + S^2) &= c_1, \quad \omega'^2 = 4\omega^3 + c^2. \end{aligned}$$

We are planning to consider the Tzitzeica type equations separately. One of the problems here is that the list of integrable fifth order evolution equations from [3] possibly is not complete.

The second question is: why we restrict ourselves by symmetries of the form $u_t = u_{xxx} + F(u, u_x, u_{xx})$ instead of general symmetries of the form

$$u_t = \Phi(u, u_x, u_{xx}, u_{xxx})? \quad (2.36)$$

The main reason is the following statement (see [12]): suppose equation (2.36) is a symmetry for equation (1.19). Then

$$\frac{d}{dy} \left(\frac{\partial \Phi(u, u_x, u_{xx}, u_{xxx})}{\partial u_{xxx}} \right) = 0.$$

Therefore, if we assume that (1.19) has no nontrivial integrals, then

$$\frac{\partial \Phi(u, u_x, u_{xx}, u_{xxx})}{\partial u_{xxx}} = \text{const.}$$

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Appendix 1. Symmetries, integrals and differential substitutions

Here an information of the integrable equations from the list (2.1)–(2.21) is presented. Only the simplest equation from the equivalence class is shown. The existence of x -integrals $J(u, u_y, u_{yy}, \dots)$ and y -integrals $I(u, u_x, u_{xx}, \dots)$ was checked till the seventh order.

Linearizing substitutions from Liouville type equations to $v_{xy} = 0$ have the form $I = v_x$ or $J = v_y$. More complicated substitutions $I = f(v_x, v_{xx}, \dots)$ or $J = g(v_y, v_{yy}, \dots)$ are presented explicitly.

Equation (2.1). The symmetries have the following form:

$$u_t = u_{xxx} - \frac{c}{2}u_x^3 - \frac{3}{2}f^2(u)u_x, \quad u_t = u_{yyy} - \frac{3u_y u_{yy}^2}{2(u_y^2 + 1)} - \frac{c}{2}u_y^3.$$

There are two the *sin*-Gordon type equations:

$$(2.1a). \quad u_{xy} = u\sqrt{u_x^2 + 1}; \quad (2.1b). \quad u_{xy} = \sin u\sqrt{u_x^2 + 1}$$

and two Liouville type equations:

$$(2.1c). \quad u_{xy} = \sqrt{u_x^2 + 1}; \quad \text{the integrals are:}$$

$$I = \frac{u_{xx}}{\sqrt{u_x^2 + 1}}, \quad J = u_{yy} - u;$$

the linearizing substitution $u_x = \sinh(y + v_x)$ gives rise to the general solution:

$$u = \int \sinh(y + f(x)) dx + g(y);$$

$$(2.1d). \quad u_{xy} = e^u \sqrt{u_x^2 + 1}; \quad \text{the integrals are:}$$

$$I = \frac{u_{xx}}{\sqrt{u_x^2 + 1}} - \sqrt{u_x^2 + 1}, \quad J = u_{yy} - \frac{1}{2}u_y^2 - \frac{1}{2}e^{2u},$$

The general solution is given by

$$u(x, y) = \ln \left(\frac{-\varphi(x)g'(y)}{(g(y) + h(x))(\varphi(x) + f(x)(g(y) + h(x)))} \right),$$

$$\varphi(x) = \exp \left(\int \frac{f(x)}{4f'(x)} dx \right), \quad h(x) = \int \frac{f'(x)\varphi(x)}{f^2(x)} dx.$$

Equation (2.2). Both x - and y -symmetries have the form (1.5), where $c_1 = c_2 = 0$. If $ab \neq 0$, then we have the *sin*-Gordon equation.

(2.2a). $u_{xy} = e^u$ is the Liouville equation. Its symmetries have the same form as for the *sin*-Gordon equation. The integrals were shown in the Introduction. The general solution

$$u(x, y) = \log \left(\frac{2f'(x)g'(y)}{(f(x) + g(y))^2} \right);$$

was found by Liouville in 1853.

Equation (2.3). The x -symmetry has the form (1.10), where $c_1 = 0, c_2 = -3/4$; the y -symmetry is of the form (1.6), where $c_1 = c_2 = 0$. It is an S-integrable equation.

Equation (2.4). Both x - and y -symmetries have the form (1.6), where $c_1 = 0, c_2 = -1/2$. It is an S-integrable equation.

Equation (2.5). The x -symmetry is of the form (1.7), the form of the y -symmetry is analogous:

$$u_t = u_{yyy} - \frac{3u_y u_{yy}^2}{2(u_y^2 + a)} - \frac{3}{2} \wp(u) u_y (u_y^2 + a).$$

If $a = 0$, then this symmetry is equivalent to (1.9).

In the general case the equation can be rewritten using the Jacobi function sn as:

$$u_{xy} = \frac{1}{\text{sn}(u, k)} \sqrt{u_x^2 + 1} \sqrt{u_y^2 + a}. \quad (2.37)$$

This is an S-integrable equation except for the degenerate cases considered below. Notice that the formulas

$$\sqrt{\wp(u, g_2, g_3) - \mu_1} = \frac{\text{cn}(u, k)}{\text{sn}(u, k)}, \quad \sqrt{\wp(u, g_2, g_3) - \mu_2} = \frac{\text{cn}(u, k)}{\text{dn}(u, k)}.$$

lead to another forms of this equation. They can be reduced to (2.37) by the substitution $(u, k) \rightarrow (\lambda u, f(k))$ (see [13], Sec. 13.22).

There are two degenerate cases for the Weierstrass function. In the first case when $\wp(u) = u^{-2}$ we have $\mu = 0$ and $\sqrt{\wp - \mu} = u^{-1}$. In the second case $\wp(u) = \sin^{-2} u - \frac{1}{3}$, $\mu = -\frac{1}{3}$ and $\sqrt{\wp - \mu} = \sin^{-1} u$.

(2.5a). Equation $u_{xy} = u^{-1} \sqrt{u_x^2 + 1} \sqrt{u_y^2 + a}$ is C-integrable, the integrals are:

$$I = \frac{u_{xx}}{\sqrt{u_x^2 + 1}} + \frac{1}{u} \sqrt{u_x^2 + 1}, \quad J = \frac{u_{yy}}{\sqrt{u_y^2 + a}} + \frac{1}{u} \sqrt{u_y^2 + a}.$$

The general solution is given by:

$$u(x, y) = \sqrt{f(x) + g(y)} \left(- \int \frac{dx}{f'(x)} - a \int \frac{dx}{g'(y)} \right)^{1/2}.$$

(2.5b). Equation $u_{xy} = (\sin u)^{-1} \sqrt{u_x^2 + 1} \sqrt{u_y^2 + a}$ is C-integrable, the integrals are:

$$I = \frac{u_{xx}}{\sqrt{u_x^2 + 1}} + \cot u \sqrt{u_x^2 + 1}, \quad J = \frac{u_{yy}}{\sqrt{u_y^2 + a}} + \cot u \sqrt{u_y^2 + a}.$$

If $a = 0$, then the general solution is

$$u(x, y) = 2 \arccos \left(\frac{f(x) + h(y) + g(y)}{2f(x)} \right)^{1/2}, \quad h(x) = \int \sqrt{f'^2 - f^2} dx.$$

If $a \neq 0$, then the general solution is

$$\begin{aligned} u(x, y) &= \arccos \Psi(x, y), \\ \Psi(x, y) &= \frac{1}{2} w(x) [e^g (\xi + h)^2 - e^{-g}] (2w' + fw) + (\xi + h)e^g, \quad g = g(y), \\ h(y) &= \int e^{-g} \sqrt{g'^2 - a} dy, \quad f'(x) = \frac{1}{2} (1 + f^2) - 2 \frac{w''}{w}, \quad \xi(x) = \int \frac{dx}{w^2(x)}. \end{aligned}$$

(2.5c). $a = 0$, $u_{xy} = f(u)u_y\sqrt{u_x^2 + 1}$. There exists the following y -integral

$$I = (u_x + \sqrt{u_x^2 + 1}) \exp(-\xi(u)), \quad \xi(u) = \int f(u) du$$

for all $f(u)$. This leads to the first order ODE:

$$u_x = \frac{1}{2} \left(h(x)e^\xi - (h(x)e^\xi)^{-1} \right).$$

All remaining equations are C-integrable. Some of them have two integrals and can be reduced to the D'Alembert equation. Others have no integrals and can be reduced to the Klein-Gordon equation.

Equation (2.6). The x -symmetry has the form (1.9) and the y -symmetry is the mKdV equation $u_t = u_{yyy} - 6u^2u_y$. The integrals are:

$$I = \frac{u_{xxx}}{u_x} - \frac{3u_{xx}^2}{2u_x^2}, \quad J = u_y - u^2.$$

The general solution is given by

$$u(x, y) = \frac{g''(y)}{2g'(y)} - \frac{g'(y)}{f(x) + g(y)}.$$

Equation (2.7). The x -symmetry has the form (1.9) and the y -symmetry is (1.10), where $c_1 = 0, c_2 = -3$. The integrals are:

$$I = \frac{u_{xxx}}{u_x} - \frac{3u_{xx}^2}{2u_x^2}, \quad J = \sqrt{u_y} - u.$$

The general solution is given by

$$u(x, y) = -\frac{g'(y)}{f(x) + g(y)} + \int \frac{(g'')^2}{4g'^2} dy.$$

Equation (2.8). The y -symmetry takes the form (1.6), where $c_1 = 0, c_2 = -1/2$ and the x -symmetry is

$$u_t = u_{xxx} - \frac{3u_{xx}^2}{2u_x} - \frac{1}{2}u_x^3.$$

This symmetry can be reduced to (1.9) by $u \rightarrow \ln u$. The integrals are:

$$I = \frac{u_{xxx}}{u_x} - \frac{3u_{xx}^2}{2u_x^2} - \frac{1}{2}u_x^2, \quad J = (u_y + \sqrt{u_y^2 + 1})e^{-u}.$$

The general solution is given by

$$u(x, y) = \ln \left[1 + \frac{g(y)}{f(x) + h(y)} \right] + \int g^{-1} \sqrt{g'^2 - g^2} dy, \quad h = -\frac{1}{2}g - \frac{1}{2} \int \sqrt{g'^2 - g^2} dy.$$

Equation (2.9). (The Goursat equation.) Both x - and y -symmetries have the form (1.11) with arbitrary constant c .

The equation is reduced to the Klein-Gordon equation $v_{xy} = \frac{1}{4}v$ by any of the following two differential substitutions:

$$(1) \ u_x = 4v_x^2, \quad u_y = v^2; \quad (2) \ u_x = v^2, \quad u_y = 4v_y^2.$$

Equation (2.10). The x -symmetry has the form (1.11), where $c = 0$ and the y -symmetry can be obtained from (1.5) by the substitution $c_2 = 0$, $u \rightarrow -\ln u$. Moreover, there exists the following second order y -symmetry $u_t = u_{yy} - 2u^{-1}(u_y^2 + au_y)$.

The integrals and the general solution are:

$$I = \frac{u_{xx}}{u_x}, \quad J = \frac{u_y + a}{u}; \quad u(x, y) = \frac{f(x) - ag(y)}{g'(y)}.$$

Equation (2.11). The x -symmetry has the form (2.27), where $q = -\frac{1}{2}$ and the y -symmetry is given by (1.5), where $c_1 = -\frac{3}{2}a^2$, $c_2 = -\frac{3}{2}b^2$. The integrals are:

$$I = \frac{u_{xxx}}{u_x} - \frac{3u_{xx}^2}{2u_x^2} - \frac{1}{2}u_x^2, \quad J = u_y - ae^u + be^{-u}.$$

In the case $a \neq 0$ the general solution is given by

$$u(x, y) = \ln g(y) + \ln \left[1 + \frac{h(y)}{f(x) - a\varphi(y)} \right], \quad \ln h = \int (ag + bg^{-1}) dy, \quad \varphi = \int gh dy;$$

if $a = 0$ then

$$u(x, y) = \ln \frac{f(x) - bg(y)}{g'(y)}.$$

Equation (2.12). The x -symmetry has the form (1.13). There are the following y -symmetries:

$$u_t = u_{yyy} - \frac{3}{2}(3 + \coth u)u_y u_{yy} + \frac{1}{4}(3 \coth^2 u + 6 \coth u + 7)u_y^3,$$

$$u_t = u_{yy} - \frac{1}{2}(3 + \coth u)u_y^2.$$

The integrals are:

$$I = \frac{e^{-u}\eta^2 - 2\eta + e^u}{\sinh u}, \quad J = \frac{u_{yy}}{u_y} - \frac{1}{2}u_y(\coth u + 3).$$

The general solution is given by:

$$u(x, y) = -\frac{1}{2} \ln(1 + \psi^2), \quad \psi = f(x)(g(y) + h(x)), \quad f' = f - \frac{1}{4}f^3 h'^2.$$

Equation (2.13). The x -symmetry has the form (1.13). There are the following y -symmetries:

$$u_t = u_{yyy} - 6u_y u_{yy} \coth u + 2(3 \coth^2 u - 1)u_y^3, \quad u_t = u_{yy} - 2u_y^2 \coth u.$$

The integrals are:

$$I = \frac{\eta - e^u}{\eta - e^{-u}}, \quad J = \frac{u_{yy}}{u_y} - 2u_y \coth u.$$

The general solution is:

$$u(x, y) = \frac{1}{2} \ln \left| \frac{\psi + 1}{\psi - 1} \right|, \quad \psi = f(x)(g(y) + h(x)), \quad h' = -\frac{f'^2 + 4f^2}{4f^3}.$$

Equation (2.14). Both x - and y -symmetries have the form (1.13). The equation is reduced to the Klein-Gordon one $v_{xy} = v$ by the following differential substitution:

$$u_x = (v^{-1}v_x \sinh u + \cosh u)^2 - 1, \quad u_y = (v^{-1}v_y \sinh u + \cosh u)^2 - 1.$$

Equation (2.15). There are x -symmetry of the form (1.12) and the following y -symmetry:

$$u_t = u_{yyy} - \frac{3u_y u_{yy}}{2u} + \frac{3u_y^3}{4u^2} - \frac{3c}{4}(2uu_{yy} + 2u_y^2 - cu^2 u_y).$$

The equation can be reduced to the Klein-Gordon equation $v_{xy} = cv$ by the following differential substitution:

$$u = v^2/z, \quad z_x = -v_x^2, \quad z_y = -cv^2.$$

If $c = 0$ then the Klein-Gordon equation is reduced to the D'Alembert equation and the following two integrals appear:

$$I = \frac{(\eta - 1)^2}{u}, \quad J = \frac{u_{yy}}{u_y} - \frac{u_y}{2u}.$$

The general solution is:

$$u(x, y) = \frac{(f(x) + g(y))^2}{z(x)}, \quad z(x) = -\int f'^2(x) dx.$$

Notice that if $c = 0$ the equation admits a second order symmetry.

Equation (2.16). There are x -symmetry of the form (1.12) and two the following y -symmetries:

$$u_t = u_{yyy} - 6u^{-1}u_y u_{yy} + 6u^{-2}u_y^3, \quad u_t = u_{yy} - 2u^{-1}u_y^2.$$

The integrals and the general solution are given by:

$$I = \frac{\eta - 1}{u}, \quad J = \frac{u_{yy}}{u_y} - 2\frac{u_y}{u}; \quad u(x, y) = \frac{f^2(x)}{h(x) + g(y)}, \quad h(x) = -\int f'^2(x) dx.$$

Equation (2.17). Both x - and y -symmetries have the form (1.12). The integrals are of the form:

$$I = \frac{u_{xx}}{\eta(\eta-1)} - \frac{2}{u}\eta(\eta-1), \quad J = \frac{u_{yy}}{\xi(\xi-1)} - \frac{2}{u}\xi(\xi-1).$$

The general solution is given by:

$$u(x, y) = \frac{(f(x) + g(y))^2}{z(x, y)}, \quad z(x, y) = - \int f'^2(x) dx - \int g'^2(y) dy.$$

Equation (2.18). There are x -symmetry of the form (1.14) and two the following y -symmetries:

$$u_t = u_{yyy} - 9u^{-1}u_y u_{yy} + 12u^{-2}u_y^3, \quad u_t = u_{yy} - 3u^{-1}u_y^2.$$

The integrals are of the form:

$$I = \frac{u_x}{u} + u^2, \quad J = \frac{u_{yy}}{u_y} - 3\frac{u_y}{u}.$$

The general solution is:

$$u(x, y) = \left(\frac{f'(x)}{2(f(x) + g(y))} \right)^{1/2}.$$

Equation (2.19). There are x -symmetry of the form (1.15) and two the following y -symmetries:

$$u_t = u_{yyy} - 3u^{-1}(2u_y + a)u_{yy} + 3au^{-2}u_y(3u_y + a) + 6u^{-2}u_y^3, \quad u_t = u_{yy} - 2u^{-1}u_y(u_y + a).$$

When $a = 0$ the y -symmetry (1.9) is also admitted. The equation can be reduced to the Klein-Gordon one $v_{xy} = -av$ by the following substitution:

$$u_x = \left(\frac{v_x}{v} - u \right) (u - \lambda), \quad u_y = \frac{1}{\lambda} \left(u \frac{v_y}{v} + a \right) (u - \lambda),$$

where λ is arbitrary parameter. If $a = 0$, then the Klein-Gordon equation is reduced to the D'Alembert equation and two integrals appear:

$$I = \frac{u_x}{u} + u, \quad J = \frac{u_{yy}}{u_y} - 2\frac{u_y}{u},$$

In this case the general solution is in the form $u(x, y) = f'(x)(f(x) + g(y))^{-1}$.

Equation (2.20). The x -symmetry is $u_t = u_{xxx} - \frac{3}{2}a u_{xx}$ and the y -symmetry has the form (1.11), where $c = 0$ and $x \rightarrow y$. The integrals are of the form:

$$I = u_{xxx} - \frac{3}{2}a u_{xx} + \frac{a^2}{2}u_x, \quad J = \frac{u_{yy}}{a u_y + \sqrt{u_y}}.$$

The general solution is given by:

$$u(x, y) = f(x) + e^{ax} \int \left(g(y) + \frac{1 - e^{-ax/2}}{a} \right) dy.$$

The limit $a \rightarrow 0$ is available here.

Equation (2.21). There are infinitely many symmetries of the form $u_t = P(\partial_x, \partial_y)u$, where P is an arbitrary polynomial with constant coefficients. In particular, there exist x - and y -symmetries of the form $u_t = P_1(\partial_x)u$ and $u_t = P_2(\partial_y)u$. If $c \neq 0$ integrals do not exist otherwise the simplest integrals are: $I = u_x$, $J = u_y$.

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